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On Harmonic Quasi-Convex Functions

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Abstract

et $\mathcal{S}_{\mathcal{H}}$ denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the unite disk $\mathbb{U} = \{z : |z| < 1\}$ where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $g(z) = \sum\limits_{k=1}^{\infty} b_k z^k$ $(|b_1| < 1)$. In this paper we introduce and study the new class of harmonic quasi-convex function. Some properties of this class are proved. Coefficient conditions, distortion bounds, extreme points, convolution conditions, convex combination for the class $\mathcal{THQ}$ are obtained .

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1 Introduction and preliminary results

A continuous functions $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$. See Clunie and Sheil-Small (see [4]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \qquad |b_1| < 1. \tag{1.1}$$

Analogous to well-known subclasses of the family $\mathcal{S}$, one can define various subclasses of the family $\mathcal{S}_{\mathcal{H}}$. A sense-preserving harmonic mapping $f \in \mathcal{S}_{\mathcal{H}}$ is in the class $\mathcal{HS}^*$ if the range $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{HS}^*$ is called a harmonic starlike mapping in $\mathbb{U}$. Likewise a function f defined in $\mathbb{U}$ belongs to the class HC if $f \in \mathcal{S}_{\mathcal{H}}$ and if $f(\mathbb{U})$ is a convex domain. A function $f \in \mathcal{HC}$ is called harmonic convex in $\mathbb{U}$. Analytically, we have

$$f \in \mathcal{HS}^* \quad \Leftrightarrow \quad \mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)}\right\} > 0, \qquad z \in \mathbb{U}.$$

$$f \in \mathcal{HC} \quad \Leftrightarrow \quad \mathrm{Re}\left\{\frac{zh''(z) + h'(z) - \overline{zg''(z) + zg'(z)}}{h'(z) - \overline{g'(z)}}\right\} > 0, \qquad z \in \mathbb{U}.$$

In [1] Noor and Thomas Introduced and studied the class of quasi-convex function Q for $f \in \mathcal{S}$.

Definition 1.1 *Let $f \in \mathcal{S}$. Then f is said to be quasi-convex in $\mathbb{U}$ if there exists a convex function g with $g(0) = 0,\ g'(0) = 1$ such that*

$$Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0, \qquad z \in \mathbb{U}.$$

Now, we define the class of harmonic quasi-convex functions denoted by $\mathcal{HQ}$.

Definition 1.2 *Let $f = h + \overline{h}$ where h and g given by (1.1). Then f is said to be harmonic quasi-convex in $\mathbb{U}$ if there exists a harmonic convex function $F = H + \overline{G}$ in $\mathbb{U}$ where*

$$H(z) = z + \sum_{k=2}^{\infty} A_k z^k, \qquad G(z) = \sum_{k=1}^{\infty} B_k z^k \qquad |B_1| < 1. \tag{1.2}$$

such that

$$Re\left\{ \frac{zh''(z) + h'(z) - \overline{zg''(z) + zg'(z)}}{H'(z) - \overline{G'(z)}} \right\} > 0, \qquad z \in \mathbb{U}. \tag{1.3}$$

it is clear that when $f(z) = F(z)$, then $\mathcal{HC} = HQ$ so that $\mathcal{HC} \subset \mathcal{HQ}$.

Note that in 1984 Clunie and Sheil-Small [4] investigated the class $\mathcal{S_H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{S_H}$ and its subclasses such that Avci and Zlotkiewicz [6], Silverman [2], Silverman and Silvia [3] , and Jahangiri [5], Ponnusamy and Kaliraj [7] studied the harmonic univalent functions.

We show first that $\mathcal{HQ} \subset \mathcal{HK}$, so that every harmonic quasi-convex functions is harmonic close-to-convex function and hence univalent in $\mathbb{U}$. To prove the following theorem we use the same technique given by [7].

Theorem 1.1 *Let $f = h + \overline{g} \in \mathcal{HQ}$, where h and g given by (1.1) and F be univalent, analytic and starlike in $\mathbb{U}$. If f satisfies*

$$Re\left\{ \frac{e^{i\theta} h'(z)}{F'(z)} \right\} > \left| \frac{g'(z)}{F'(z)} \right|, \qquad z \in \mathbb{U}, \tag{1.4}$$

then f is sen-preserving harmonic and close-to-convex $\mathcal{HK}$ and hence univalent in $\mathbb{U}$.

Proof. Let $T = h + \epsilon g \in \mathcal{HQ}$ where $|\epsilon| = 1$. By (1.4) it follows that

$$\mathrm{Re}\left\{ \frac{e^{i\theta} T'(z)}{F'(z)} \right\} = \mathrm{Re}\left\{ \frac{e^{i\theta} h'(z)}{F'(z)} \right\} + \mathrm{Re}\left\{ \frac{\epsilon e^{i\theta} g'(z)}{F'(z)} \right\} \geq \mathrm{Re}\left\{ \frac{e^{i\theta} h'(z)}{F'(z)} \right\} - \left| \frac{g'(z)}{F'(z)} \right| > 0, \qquad z \in \mathbb{U}.$$

Moreover, by (1.5), we see that

$$\left| \frac{g'(z)}{F'(z)} \right| < \mathrm{Re}\left\{ \frac{e^{i\theta} h'(z)}{F'(z)} \right\} \leq \mathrm{Re}\left| \frac{e^{i\theta} h'(z)}{F'(z)} \right| = \left| \frac{h'(z)}{F'(z)} \right|,$$

So that (as $F'(z) \neq 0$), $|g'(z)| < |h'(z)|$. Thus from the classical analytic characterization for close-to-convex function [[4] Theorem 2.17], we obtain that $T = h + \epsilon g$ is close -to-convex in $\mathbb{U}$ for each ϵ with $|\epsilon| = 1$. By Theorem 1.3, we obtain that f is close-to-convex and hence univalent in $\mathbb{U}$.

We further denote by $\mathcal{THQ}$ the subclass of $\mathcal{HQ}$ such that the functions $f = h + \overline{g}$ where h and g of the form:

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.5}$$

and

$$F(z) = H(z) + \overline{G_n(z)} = z - \sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k}. \tag{1.6}$$

In the following, we will give the sufficient condition for functions $f = h + \overline{g}$ where h and g given by (1.1) to be in the class $\mathcal{HQ}$ and it is shown that these coefficient condition is also necessary for functions in the class $\mathcal{THQ}$. Also, we obtain distortion theorems and characterize the extreme points for functions in $\mathcal{THQ}$. Convolution and closure theorems are also obtained.

2 Coefficient Bounds

We begin with a sufficient coefficient condition for functions in $\mathcal{HQ}$.

Theorem 2.1 *Let $f = h + \overline{g}$ where h and g given by (1.1) and $F = H + \overline{G}$ given by (1.2). If*

$$\sum_{n=1}^{\infty} n^2[|a_n| + b_n|] \leq 2, \tag{2.1}$$

where $a_1 = 1$, then f is sense-preserving, harmonic univalent in $\mathbb{U}$, and $f \in \mathcal{HQ}$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| = 1 - \left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right| \\ &> 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|} \geq 1 - \frac{\sum_{n=1}^{\infty} n^2|b_n|}{1 - \sum_{n=2}^{\infty} n^2|a_n|} \geq 0,\end{aligned}$$

which proves univalence. Note that f is sense-preserving in $\mathbb{U}$. This is because

$$\begin{aligned}|h'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n^2|a_n| \geq \sum_{n=1}^{\infty} n^2|b_n| \\ &> \sum_{n=1}^{\infty} n^2|b_n||z|^{n-1} \geq \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.\end{aligned}$$

From (1.3), let $w(z) = \frac{(zf'(z))'}{F'(z)}$ and by using the fact that $\operatorname{Re} w > 0$ if and only if $|1+w| \geq |1-w|$, it suffices to show that

$$\left|F'(z) + (zf'(z))'\right| - \left|F'(z) - (zf'(z))'\right| \geq 0. \tag{2.2}$$

Substituting for $f(z)$ and $F(z)$ given by (1.1) and (1.2), respectively in (2.2) yields, by (2.1) we obtain

$$\begin{aligned}&\left|F'(z) + (zf'(z))'\right| - \left|F'(z) - (zf'(z))'\right| \\ = \;&\left|2 + \sum_{n=2}^{\infty}[n^2a_n + nA_n]z^{n-1} + \overline{\sum_{n=1}^{\infty}[n^2b_n - nB_n]z^{n-1}}\right| \\ &-\left|\sum_{n=2}^{\infty}[n^2a_n - nA_n]z^{n-1} + \overline{\sum_{n=1}^{\infty}[n^2b_n + nB_n]z^{n-1}}\right| \\ \geq\; & 2 - \sum_{n=2}^{\infty}[n^2|a_n| + n|A_n|]|z|^{n-1} - \sum_{n=1}^{\infty}[n^2|b_n| - n|B_n|]|z|^{n-1} \\ &- \sum_{n=2}^{\infty}[n^2|a_n| - n|A_n|]|z|^{n-1} - \sum_{n=1}^{\infty}[n^2|b_n| + n|B_n|]|z|^{n-1} \\ \geq\; & 2\left\{1 - \sum_{n=2}^{\infty} n^2|a_n||z|^{n-1} - \sum_{n=1}^{\infty} n^2|b_n||z|^{n-1}\right\} \\ \geq\; & 2\left\{1 - \sum_{n=2}^{\infty} n^2|a_n| - \sum_{n=1}^{\infty} n^2|b_n|\right\}.\end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is complete.

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{n^2} x_n z^n + \sum_{n=1}^{\infty} \frac{1}{n^2} \overline{y_n z^n} \tag{2.3}$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $\mathcal{HQ}$ because

$$\sum_{n=1}^{\infty} n^2[|a_n| + |b_b|] = 1 + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2.$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f = h + \overline{g}$ where h and g are of the form (1.5).

Theorem 2.2 *Let $f = h + \overline{g}$ be given by (1.5) and $F = H + \overline{G}$ given by (1.6). Then $f \in \mathcal{THQ}$, if and only if*

$$\sum_{n=1}^{\infty} n^2[|a_n| + |b_n|] \leq 2. \tag{2.4}$$

Proof. Since $\mathcal{THQ} \subset \mathcal{HQ}$, we only need to prove the "only if" part of the theorem. To this end, for functions f and F given by (1.5) and (1.6), respectively , we notice that the condition (1.3) is equivalent to

$$\text{Re}\left\{ \frac{1 - \sum_{n=2}^{\infty} n^2 a_n z^{n-1} - \sum_{n=1}^{\infty} n^2 b_n \overline{z^{n-1}}}{1 - \sum_{n=2}^{\infty} n A_n z^{n-1} + \sum_{n=1}^{\infty} B_n \overline{z^{n-1}}} \right\} > 0. \tag{2.5}$$

The above required condition (2.5) must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{1 - \sum_{n=2}^{\infty} n^2 |a_n| r^{n-1} - \sum_{n=1}^{\infty} n^2 |b_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} n |A_n| r^{n-1} + \sum_{n=1}^{\infty} |B_n| r^{n-1}} > 0. \tag{2.6}$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.4) is negative. This contradicts the required condition for $f \in \mathcal{THQ}$ and so the proof is complete.

3 Distortion Bounds and Extreme Points.

In this section, first we shall obtain distortion bounds for functions in $\mathcal{THQ}$.

Theorem 3.1 *If $f \in \mathcal{THQ}$. Then for $|z| = r < 1$ we have*

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{4}\Big(1 - |b_1|\Big) r^2,$$

and

$$|f(z)| \geq (1 - |b_1|)r - \frac{1}{4}\Big(1 - |b_1|\Big) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in \mathcal{THQ}$. Taking the absolute value of f we obtain

$$\begin{aligned}
|f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n\right| \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \\
&\leq (1+|b_1|)r + r^2 \sum_{n=2}^{\infty} [|a_n| + |b_n|]) \\
&\leq (1+|b_1|)r + \frac{1}{4}\Big(\sum_{n=2}^{\infty} 2^2[|a_n| + |b_n|]\Big) r^2 \\
&\leq (1+|b_1|)r + \frac{1}{4}\Big(\sum_{n=2}^{\infty} n^2[|a_n| + |b_n|]\Big) r^2 \\
&\leq (1+|b_1|)r + \frac{1}{4}\Big(1 - |b_1|\Big) r^2,
\end{aligned}$$

for $|b_1| < 1$ show that the bounds given Theorem 3.1 are sharp.

The following covering result follows from the left hand inequality in Theorem 3.1.

Corollary 3.2 *If If $f \in \mathcal{THQ}$. Then*

$$\Big\{w : |w| < \frac{3}{4}(1 - |b_1|)\Big\} \subset f(\mathbb{U}).$$

Next we determine the extreme points of closed convex hulls of $\mathcal{THQ}$ denoted by $clco\,\mathcal{THQ}$.

Theorem 3.3 *$f \in clco\,\mathcal{THQ}$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)) \tag{3.1}$$

where $h_1(z) = z$, $h_n(z) = z - \frac{1}{n^2} z^n$ $(n = 2, 3, ...)$, $g_n(z) = z + \frac{1}{n^2}\overline{z}^n$ $(n = 1, 2, 3, ..)$, $\sum_{n=1}^{\infty} (X_n + Y_n) = 1$, $X_n \geq 0$, $Y_n \geq 0$. In particular, the extreme points of $\mathcal{THQ}$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions f of the form (3.1) we have

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)) \\
&= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1}{n^2} X_n z^n + \sum_{n=1}^{\infty} \frac{1}{n^2} Y_n \overline{z}^n
\end{aligned}$$

Then

$$\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1,$$

and so $f \in clco\,\mathcal{THQ}$.

Conversely, suppose that $f \in clco\,\mathcal{THQ}$. Setting

$$\begin{aligned}
&X_n = n^2 |a_n|\, 0 \leq X_n \leq 1 \ (n = 2, 3, ...), \\
&Y_n = n^2 |b_n|\, 0 \leq Y_n \leq 1 \ (n = 1, 2, 3, ...),
\end{aligned}$$

and $X_1 = 1 - \sum\limits_{n=2}^{\infty} X_n - \sum\limits_{n=1}^{\infty} Y_n$. Therefore, f can be written as

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} \frac{X_n}{n^2} z^n + \sum_{n=1}^{\infty} \frac{Y_n}{n^2} \overline{z}^n \\ &= z + \sum_{n=2}^{\infty} (h_n(z) - z) X_n + \sum_{n=1}^{\infty} (g_n(z) - z) Y_n \\ &= \sum_{n=2}^{\infty} h_n(z) X_n + \sum_{n=1}^{\infty} g_n(z) Y_n + z \left(1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n \right) \\ &= \sum_{n=1}^{\infty} (h_n(z) X_n + g_n(z) Y_n), \text{ as required.} \end{aligned}$$

4 Convolution and Convex Combination.

In this section, we show that the class $\mathcal{THQ}$ is invariant under convolution and convex combination of its member.

For harmonic functions $f(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=1}^{\infty} b_n \overline{z}^n$ and $\Omega(z) = z - \sum\limits_{n=2}^{\infty} \psi_n z^n + \sum\limits_{n=1}^{\infty} \phi_n \overline{z}^n$ the convolution of f and F is given by

$$(f * \Omega)(z) = f(z) * \Omega(z) = z - \sum_{n=2}^{\infty} a_n \psi_n z^n + \sum_{n=1}^{\infty} b_n \phi_n \overline{z}^n. \tag{4.1}$$

Theorem 4.1 *Let $f \in \mathcal{THQ}$ and $\Omega \in \mathcal{THQ}$. Then $f * \Omega \in \mathcal{THQ}$.*

Proof. Then the convolution $f * \Omega$ is given by (4.1). We wish to show that the coefficients of $f * \Omega$ satisfy the required condition given in Theorem 2.2. For $\Omega \in \mathcal{THQ}$ we note that $|\psi_n| \leq 1$ and $|\phi_n| \leq 1$. Now, for the convolution function $f * \Omega$, we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} n^2 |a_n| |\psi_n| + \sum_{k=1}^{\infty} n^2 |b_n| |\phi_n| \\ \leq & \sum_{n=2}^{\infty} n^2 |a_n| + \sum_{k=1}^{\infty} n^2 |b_n| \leq 1. \end{aligned}$$

Therefore $f * \Omega \in \mathcal{THQ}$.

We now examine the convex combination of $\mathcal{THQ}$.

Let the functions $f_j(z)$ be defined , for $j = 1, 2, ...$ by

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n,j}| z^n + \sum_{n=1}^{\infty} |b_{n,j}| \overline{z}^n. \tag{4.2}$$

Theorem 4.2 *Let the functions $f_j(z)$ defined by (4.2) be in the class $\mathcal{THQ}$ for every $j = 1, 2, ..., m$. Then the functions $t_j(z)$ defined by*

$$t_j(z) = \sum_{j=1}^{m} c_j f_j(z), \quad (0 \leq c_j \leq 1) \tag{4.3}$$

is also in the class $\mathcal{THQ}$ where $\sum\limits_{j=1}^{m} c_j = 1$.

Proof. According to the definition of t_j, we can write

$$t_j(z) = z - \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} c_j a_{n,j} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} c_j b_{n,j} \right) \overline{z}^k \tag{4.4}$$

Further , since $f_j(z)$ are in $\mathcal{THQ}$ for every $(j = 1, 2, ...)$. Then by (2.4) we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \left\{ n^2 \Big(\sum_{j=1}^{m} c_j [|a_{k,j}||b_{k,j}|] \Big) \right\} \\ = & \sum_{j=1}^{m} c_j \left(\sum_{n=1}^{\infty} n^2 [|a_{n,j}| + |b_{n,j}|] \right) \\ \leq & \sum_{j=1}^{m} c_j 2 \leq 2. \end{aligned}$$

Hence the theorem follows .

Corollary 4.3 *The class $\mathcal{THQ}$ is close under convex linear combination.*

Proof. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $\mathcal{THQ}$. Then the function $\Psi(z)$ defined by

$$\Psi(z) = \mu f_1(z) + (1 - \mu) f_2(z) \qquad (0 \leq \mu \leq 1)$$

is in the class $\mathcal{THQ}$. Also, by taking $m = 2$, $t_1 = \mu$ and $t_2 = (1 - \mu)$ in Theorem 4.1, we have the corollary.

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